On efficient presentations for infinite families of 2-groups with fixed coclass

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Abstract

Almost all 2-groups of a fixed coclass fall into finitely many infinite families so that all groups of a family can be described by a single parametrised presentation. Our aim is to investigate whether the infinite families of groups with trivial Schur multiplicator have parametrised presentations of deficiency zero. We describe a general approach towards this aim. As application, we determine parametrised deficiency zero presentations for some infinite families and provide experimental evidence for such presentations in other cases. In particular, we obtain 6 infinite families of 3-generator groups for which we exhibit conjectural descriptions of parametrised deficiency zero presentations.

1 Introduction

The construction of short or efficient presentations for a finite group is a central problem in group theory. Such presentations yield highly compact descriptions of the considered group. They are usually difficult to determine. We refer to [5] for a general introduction on group presentations and to [7] for the algorithmic theory.

A measure for the size of a presentation is the *deficiency*: for a presentation $\langle X \mid R \rangle$ this is defined as $|X| - |R|$ and the deficiency $\mathrm{def}(G)$ of a group G is the maximum of the deficiencies of its finite presentations. It is well-known that the deficiency of a finite group is non-positive. Of particular interest is the boundary case: the finite groups of deficiency zero.

There are various examples of finite groups with deficiency zero known, but a systematic structure theory is not available. Also, there is no algorithm to determine the deficiency of a finite group. We refer to [4] for many examples of deficiency zero groups and further background.

Our aim is to construct efficient presentations for finite 2-groups using the ideas of coclass theory. In [2] a construction for infinite families of p-groups with fixed coclass is introduced and it is proved that each infinite family $\mathcal{G} = (G_0, G_1, \ldots)$ can be described by a single parametrised presentation $\langle X \mid R_j \rangle$ such that $G_j \cong \langle X \mid R_j \rangle$ for $j \in \mathbb{N}_0$. We recall the explicit construction in more detail below.

The relations R_j of these parametrised presentation depend on j, but the number $|R_j|$ is independent of j. We define the deficiency of such a parametrised presentation $\langle X \mid R_j \rangle$ by $|X| - |R_j|$. The deficiency $\mathrm{def}(\mathcal{G})$ of the infinite family is then defined as the maximum of the deficiencies of the finite parametrised presentations for $\mathcal{G}$. Note that $\mathrm{def}(\mathcal{G})$ is a lower bound for the deficiency of every group in the family $\mathcal{G}$. In particular, if an infinite family has deficiency zero, then every group in the family has deficiency zero.

In this paper we describe a general approach towards constructing infinite families of 2-groups with deficiency zero. Every such infinite family is associated with an infinite pro-2-group of finite coclass with trivial Schur multiplicator. Our first step is the construction of such infinite pro-2-groups together with deficiency zero presentations for them. Then given such a pro-2-group, we use experimental methods to construct the associated infinite families up to isomorphism. We describe two approaches to determine short parametrised presentations for all infinite families. We then try to shorten the obtained parametrised presentations for each individual infinite family further using a combination of theoretical and experimental methods. Our approach often succeeds in determining or conjecturing a parametrised deficiency zero presentation.

We apply our general approach to various infinite families of 2-groups with trivial Schur multiplicator. For some of them we determine a parametrised deficiency zero presentation. For others we use experimental evidence to conjecture such presentations.

Problem 5 in [4] asks whether there are infinitely many 3-generator p-groups with deficiency zero. A positive solution for this problem is given in [3]. Our results indicate a general approach towards constructing infinite families of 3-generator 2-groups with deficiency zero. As examples, we determine 6 infinite families of 3-generator 2-groups for which we conjecture that they have deficiency zero.

We note that the situation is significantly different for odd p. By [1] there are at most finitely many p-groups with trivial Schur multiplicator of a given coclass for odd p. This implies that our approach will not yield any infinite family of p-groups with deficiency zero for any odd prime p.

2 Coclass theory

In this section we recall some of the fundamental features of coclass theory as far as we need them later. In particular, we recall the construction of the infinite families associated with coclass theory.

2.1 Infinite pro-p-groups of finite coclass

The structure of the infinite pro-p-groups of finite coclass has been studied extensively and it is well-understood. We recall some of its essential features in this section and refer to [6] for further information and proofs.

Let S be an infinite pro-p-group of finite coclass r. Then the lower central series quotients $S_i := S/\gamma_i(S)$ are finite p-groups. The group S can be constructed as the inverse limit of the groups S_i. Further, there exists a $j \in \mathbb{N}$ so that $\gamma_j(S) \cong \mathbb{Z}_p^d$, where $\mathbb{Z}_p$ denotes the p-adic numbers and $d \in \mathbb{N}$, and S_j has coclass r. We fix one such j and denote $T := \gamma_j(S)$ as *translation subgroup* of S with *point group* $P := S/T$. We usually use additive notation for T and multiplicative notation for the lower central series members $\gamma_j(S)$.

The point group P acts *uniserially* on the translation subgroup T; that is, the series defined by $T_0 = T$ and $T_{i+1} = [T_i, S]$ for $i \geq 1$ satisfies $[T_i : T_{i+1}] = p$ for all $i \in \mathbb{N}_0$ and the subgroups $T_0, T_1, \ldots$ are the only P-invariant subgroups in T. This implies that $pT_i = T_{i+d}$, since pT_i is P-invariant of index p^d in T_i.

If T was chosen as $T = \gamma_j(S)$ for j, then every subgroup $\gamma_{j+i}(S)$ for $i \in \mathbb{N}_0$ is also a possible choice for T. Thus there are infinitely many possible choices for T. All these choices have the same rank d over $\mathbb{Z}_p$ and we call d the *dimension* of S.

2.2 Infinite families of p-groups

We briefly recall the construction in [2] of the groups in an infinite family. Throughout, let S be a pro-p-group of finite coclass r and dimension d. As a first step, we choose a translation subgroup T for S of the form $T = \gamma_l(S)$ for some large enough $l \in \mathbb{N}$; see [2] for explicit bounds on l. Let P be the point group corresponding to T.

As shown in [2], there exists an integer f so that for every $i \geq f$ the well-known natural sequence of cohomology groups reduces to the split short exact sequence

$$0 \to H^2(P, T) \to H^2(P, T/T_i) \to H^3(P, T_i) \to 0.$$

This induces an isomorphism $\rho_i : H^2(P, T/T_i) \to H^2(P, T) \oplus H^3(P, T_i)$. To simplify notation we also just write ρ for ρ_i. For explicit bounds on f we refer to [2]. The multiplication by p induces a P-module isomorphism $T_i \to T_{i+d}$ and thus induces an isomorphism

$$\mu_i : H^3(P, T_i) \to H^3(P, T_{i+d}).$$

Again, we simply write μ for μ_i.

Definition 1 *Let $\beta \in H^3(P, T_e)$ for some $e \in \{f, \ldots, f + d - 1\}$ and $\epsilon \in H^2(P, T)$ be an element which defines S as extension of P by T. Define G_j as the extension of P by T/T_{e+jd} via the cocycle $\rho^{-1}(\epsilon \oplus \mu^j(\beta)) \in H^2(P, T/T_{e+jd})$. Then $\mathcal{G}_\beta = (G_0, G_1, \ldots)$ is the* infinite family *associated with β.*

Note that $H^3(P, T_e)$ is a finite group, as P is finite. Hence we obtain finitely many infinite families by this construction. Different β's can lead to families with isomorphic groups. Thus our description for the construction of infinite families yields a complete, but not irredundant list.

Remark 2 *G_j is a group of order $p^{l+e+r+jd}$, class $l + e + jd$ and coclass r. Note that G_{j+1} is not a quotient of G_j, unless $\beta = 0$.*

The d families of groups arising for $\beta = 0$ play a special role. They are called the *main line families*. Their groups are the lower central series quotients of S.

We say that k is the *distance* of a group G from the main line, if k is minimal with $G/\gamma_{cl(G)+1-k}(G)$ a main line group. Thus the main line groups have distance 0 from the main line, while all other groups in infinite families have positive distance. The following allows to define the distance of an infinite family from the main line as the distance of the individual groups.

Remark 3 *All groups in an infinite family have the same distance from the main line.*

We consider a relation between different families. Let $\gamma(G)$ denote the last non-trivial subgroup in the lower central series of G.

Remark 4 *If $\mathcal{G} = (G_0, \ldots)$ is an infinite family, then $\mathcal{G}/\gamma = (G_0/\gamma(G_0), \ldots)$ is also an infinite family. Further, If $\mathcal{G}$ has distance $k > 1$, then $\mathcal{G}/\gamma$ has distance $k - 1$.*

3 The Schur multiplicator

The Schur multiplicator $M(G)$ of a finite group G yields a useful first invariant in the search for short presentations of groups: Schur proved that $M(G)$ is a finite abelian group which is generated by $-\mathrm{def}(G)$ elements. Hence finite groups with deficiency zero have trivial Schur multiplicator. Swan [8] proved that the converse does not hold by constructing an explicit counterexample. It is still open to determine a counterexample among the groups of prime-power order.

Lemma 5 *Let $\mathcal{G}$ be an infinite family of deficiency zero. Then $p = 2$ and the associated pro-p-group S has trivial Schur multiplicator and is generated by 2 or 3 elements.*

Proof: If the infinite family has deficiency zero, then every group in the family has deficiency zero and thus trivial Schur multiplicator. By [1] this implies that $p = 2$ and $M(S) = 1$. The theorem by Golod and Shafarevic yields, that finite groups of deficiency zero have at most 3 generators. •

4 Parametrised presentations

It is well-known how to construct presentations for extensions from presentations of the individual groups. Here we extend this process to determine parametrised presentations for the groups in an infinite family $\mathcal{G} = (G_0, G_1, \ldots)$ with underlying pro-p-group S. Let $\langle X \mid R\rangle$ denote a finite pro-p-presentation for the group S. Let $X = \{g_1, \ldots, g_n\}$ and $R = \{r_1, \ldots, r_m\}$, where $r_1, \ldots, r_m$ are words in X. We first consider the case of the main line families.

Lemma 6 *If $\mathcal{G} = (G_0, G_1, \ldots)$ is a main line family associated to S, then there exists a word t in X such that $G_j \cong \langle X \mid R \cup \{t^{p^j}\}\rangle$ for $j \in \mathbb{N}_0$.*

Proof: For $j \in \mathbb{N}_0$, the group G_j is a quotient of S; that is, $G_j \cong S/\gamma_{l+e+jd+1}(S)$. The subgroup $\gamma_{l+e+jd+1}(S)$ is a uniserial S-module and thus generated as normal subgroup in S by a single element t_j, say. As $\gamma_{l+e+(j+1)d+1}(S) = \gamma_{l+e+jd+1}(S)^p$, we can choose $t_{j+1} = t_j^p$ and thus $t_{j+1} = t_1^{p^j}$. Finally, $t = t_1$ can be written as an element in X and thus the result follows. •

This implies the following corollary.

Corollary 7 *If $\mathcal{G}$ is a main line family associated with S, then $0 > \mathrm{def}(\mathcal{G}) \geq \mathrm{def}(S) - 1$. In particular, if $\mathrm{def}(S) = 0$, then $\mathrm{def}(\mathcal{G}) = -1$.*

Proof: By [1], the main line groups have a non-trivial Schur multiplicator and hence they have negative deficiency. By Lemma 6, $\mathrm{def}(\mathcal{G}) \geq \mathrm{def}(S) - 1$. •

Now we consider the general case of arbitrary families associated to S.

Theorem 8 *Let $\mathcal{G} = (G_0, G_1, \ldots)$ be an infinite family associated to the infinite pro-p-group S. Let $Y = \{t_1, \ldots, t_d\}$ be a set of d abstract generators. Then the family has a parametrised presentation so that for $j \in \mathbb{N}_0$*

$$G_j \cong \langle X \cup Y \mid R_j \cup Q_j \cup O \rangle$$

with

- $R_j = \{r_i(X)y_i(Y)^{p^j}, t(X)t_1^{-1}y_t(Y)^{p^j} \mid 1 \leq i \leq m\}$ *for some words* $y_1, \ldots, y_m, t, y_t$,
- $Q_j = \{t_1 t_i^{c_i(X)}, t_1^{p^{b+j}}, [t_1, t_i] \mid 2 \leq i \leq d\}$ *for some words* $c_2, \ldots, c_d$, *and*
- $O = \{t_1^{g_h} v_{1,h}(Y) \mid 1 \leq h \leq n\}$.

Proof: The group G_j defined as an extension of $P = S/T$ by T/T_{e+jd} for $T = \gamma_l(S)$. As a first step we note that the construction from [2] is not affected by increasing l. Hence we adjust l (and thus e) suitably so that $e = bd$ for some $b \in \mathbb{N}$. This is not strictly necessary, but it simplifies the outline of the remaining proof.
By Lemma 6, the group P has a presentation $P = \langle X \mid R \cup \{t\} \rangle$ for some word t in X. Further, the group T/T_{e+jd} is generated by d elements $Y = \{t_1, \ldots, t_d\}$, say. The uniserial action of P on T implies that we can choose the generators for T/T_{e+jd} so that $t_{i+1} = t_i^{g_i}$ for some $g_i \in P$ for $1 \leq i \leq d-1$. Hence all generators are conjugate under the action of the point group P.
Every extension of P by T/T_{e+jd} has a presentation on $X \cup Y$ with relations of the form

$$\begin{array}{rcl} r_i(X) & = & w_{i,j}(Y) \text{ for } 1 \leq i \leq m, \\ t(X) & = & w_{t,j}(Y), \\ t_i^{g_h} & = & v_{i,h,j}(Y) \text{ for } 1 \leq i \leq d, 1 \leq h \leq n \\ t_i^{p^{b+j}} & = & 1 \text{ for } 1 \leq i \leq d, \\ [t_h, t_i] & = & 1 \text{ for } 1 \leq h < i \leq d, \end{array}$$

for certain words $w_{i,j}, w_{t,j}$ and $v_{i,h,j}$. We prove that such a presentation can be transformed into the desired presentation.
1. Step. We show that $w_{i,j}(Y) = y_i(Y)^{-p_j}$ for $1 \leq i \leq m$. The extension G_j arises from a cocycle $\epsilon \oplus \mu^j(\beta)$. The cocycle ϵ defines the main line family as extension. For the main line family we can choose $w_{i,j} = 1$. Thus the words $w_{i,j}$ for $1 \leq i \leq m$ depend on β only. As μ is multiplication with p, it follows that the corresponding words $w_{i,j}$ for G_0 get powered by p when j increases. Hence the result follows.
2. Step. A similar argument as in 1. yields that $w_{t,j}(Y) = t_1 y_t(Y)^{-p^j}$ for some word y_t.
3. Step. The words $v_{i,h,j}$ describe the action of the point group P on the module T/T_{e+jd}. By [2], this action is induced by the action of P on T and hence we can choose the words $v_{i,h,j}$ independent of j.
4. Step. It sufficient to use $i = 1$ in the 3rd, 4th and 5th type of relation, since $t_2, \ldots, t_d$ are all conjugate to t_1 under the action of P. We add the relations $t_1 t_i^{c_i(X)}$ to express this conjugacy relation. •

The following theorem yields an improvement on Theorem 8 for families which are close to the main line.

Theorem 9 *Let $\mathcal{G} = (G_0, \ldots)$ be an infinite family associated to the infinite pro-p-group S. If $\mathcal{G}/\gamma$ has the parametrised presentation $\langle X \mid R_j \rangle$, then $\mathcal{G}$ has a parametrised presentation $\langle X \mid R'_j \cup \{s^{p^j}\}\rangle$, where s is a word in X and $R'_j = \{rs^{e_r} \mid r \in R_j\}$ for certain $e_r \in \mathbb{Z}$.*

Proof: Every group G_j is an extension of $G_j/\gamma(G_j)$ by $\gamma_j(G_j)$. The group $\gamma_j(G_j)$ is cyclic of order p and thus generated by s_j, say. Using similar arguments as in the proof of Theorem 8, we observe that $s_j = s_1^{p^j}$. Further, we note that $s_j \in \gamma(G_j) \leq \Phi(G_j)$ and thus s_j and in particular $s = s_1$ can be written as a word in X. It remains to observe that the exponents e_r are independent of j. This, again, follows from the construction of the groups in the infinite families using cocycles $\epsilon \oplus \mu^j(\beta)$ as for the proof of Theorem 8. •

Theorems 8 and 9 now imply the following.

Corollary 10 *Let $\mathcal{G}$ be a family associated with S of distance k to the main line.*
a) $\mathrm{def}(\mathcal{G}) \geq \mathrm{def}(S) - (d + n)$.
b) $\mathrm{def}(\mathcal{G}) \geq \mathrm{def}(S) - (k + 1)$.

Proof: a) This follows directly from counting the generators and relations in Theorem 8.
b) We use induction on k. If $k = 0$, then $\mathcal{G}$ is a main line family and the result follows from Lemma 6. Suppose that $\mathcal{G}/\gamma$ has distance $k - 1$. Then $\mathrm{def}(\mathcal{G}/\gamma) \geq \mathrm{def}(S) - k$ by induction. Theorem 9 implies that $\mathrm{def}(\mathcal{G}) \geq \mathrm{def}(\mathcal{G}/\gamma) - 1 \geq \mathrm{def}(S) - k - 1$ as desired. •

We believe that a much closer connection between the deficiencies of an infinite family and its associated pro-p-group holds. We propose the following conjecture.

Conjecture 11 *If $\mathcal{G}$ is an infinite family of deficiency zero, then S is a pro-2-group of deficiency zero.*

5 Pro-2-groups with trivial Schur multiplicator

The construction of the infinite families of 2-groups of deficiency zero requires as a first step the determination of the infinite pro-2-groups with trivial Schur multiplicator. Further, as described in Section 4, it is useful to determine short presentations for these infinite pro-2-groups.

5.1 Coclass at most 3

By [1], there are 5 infinite pro-2-groups of coclass at most 3 and trivial Schur multiplicator. Pro-2-presentations of these groups are given in the following.

- $S_1 := \langle a, t \mid a^2, t^a/t^{-1}\rangle$.
- $S_2 := \langle a, t \mid a^4, t^a/t^{-1}\rangle$.
- $S_3 := \langle a, t \mid a^8, t^a/t^{-1}\rangle$.
- $S_4 := \langle a, t, b \mid a^2/b^2, b^4, t^a/(t^{-1}b), b^t/b^{-1}, b^a/b^{-1}\rangle$.
- $S_5 := \langle a, b, c, t_1, t_2, d \mid a^2/d, b^2/t_2, c^2/t_1, d^2, b^a/c, c^a/b, t_1^a/t_2, t_2^a/t_1, c^b/(ct_1^{-1}t_2d), t_1^b/t_1^{-1}, t_2^b/t_2, t_1^c/t_1, t_2^c/t_2^{-1}, (d \text{ central})\rangle$.

The following table lists some properties of these groups.

group	coclass	dimension	abelian invariants	metacyclic
S_1	1	1	$(2,2)$	yes
S_2	2	1	$(2,4)$	yes
S_3	3	1	$(2,8)$	yes
S_4	3	1	$(2,4)$	no
S_5	3	2	$(2,4)$	no

The abelian invariants exhibit that all these groups are 2-generated groups. Note that S_1, S_2 and S_3 have deficiency zero by their defining presentations. We consider the deficiency of S_4 and S_5 in the following.

Lemma 12 *S_4 is an extension of the cyclic group of order 4 by S_1. Further,*

$$S_4 = \langle a, u \mid a^2/u^4, (u^2)^a/u^{-2} \rangle$$

and thus S_4 has deficiency zero.

Proof: The defining presentation of S_4 exhibits that the subgroup $\langle b \rangle$ of S_4 is a cyclic normal subgroup of order 4 and its quotient is isomorphic to S_1. Let $u = at^{-1}$ and replace t by u in the defining presentation. Then

$$\begin{aligned} S_4 &= \langle a, b, u \mid a^2 = b^2, b^4 = 1, a^2 = u^2 b, b^{u^{-1}a} = b^{-1}, b^a = b^{-1} \rangle, \\ &= \langle a, b, u \mid a^2 = b^2, b^4 = 1, b = u^2, b^{u^{-1}a} = b^{-1}, b^a = b^{-1} \rangle. \end{aligned}$$

Now we eliminate b using that $b = u^2$ and find

$$S_4 = \langle a, u \mid a^2 = u^4, u^8 = 1, (u^2)^a = u^{-2} \rangle.$$

Note that $(u^2)^a = u^{-2}$ implies that $(u^4)^a = ((u^2)^a)^2 = u^{-4}$ and $a^2 = u^4$ implies that $(u^4)^a = u^4$ holds. Hence $u^8 = 1$ follows from the other relations and thus we finally obtain the desired presentation. •

Lemma 13 *S_5 is an extension of the cyclic group of order 2 by an infinite pro-2-group of dimension 2. Further,*

$$S_5 = \langle a, b \mid [b, a^2], a^2/[b, a]^2, (b^2)^{[b,a]}/b^{-2} \rangle$$

and hence S_5 has deficiency at least -1.

Proof: The defining presentation of S_5 exhibits that $D = \langle d \rangle$ is a central subgroup of order 2 in S_5. Its quotient is a uniserial 2-adic space group of dimension 2 with translation subgroup $T/D = \langle t_1, t_2 \rangle D/D$ and point group $\langle a, b, c \rangle T/T$ of order 8. We eliminate the generators t_1, t_2 and d in the defining presentation of S_5 using that $d = a^2$, $t_1 = c^2$ and $t_2 = b^2$. This yields

$$\begin{aligned} S_5 = \langle a, b, c \quad \mid \quad & a^4 = 1, b^a = c, c^a = b, c^b = c^{-1} b^2 a^2, \\ & (c^2)^b = c^{-2}, (b^2)^c = b^{-2}, (a^2)^b = a^2, (a^2)^c = a^2 \rangle. \end{aligned}$$

Note that the centrality of a^2 follows already from the relations $b^a = c$ and $c^a = b$. Thus the relations $(a^2)^b = a^2$ and $(a^2)^c = a^2$ are redundant. Further, the relation $(c^2)^b = c^{-2}$ is redundant, since $c^b = c^{-1}b^2a^2$ and $(b^2)^c = b^{-2}$. Now we substitute c by $u = b^{-1}c$ and obtain that

$$S_5 = \langle a, b, u \mid a^4 = 1, b^a = bu, u^a = u^{-1}, a^2 = u^2, (b^2)^u = b^{-2}\rangle.$$

The relation $a^4 = 1$ is redundant in this presentation, as $u^a = u^{-1}$ and $a^2 = u^2$ holds. Thus

$$S_5 = \langle a, b, u \mid b^a = bu, u^a = u^{-1}, a^2 = u^2, (b^2)^u = b^{-2}\rangle.$$

Finally, we eliminate u using that $u = [b, a]$ and note that $[b, a]^a = [b, a]^{-1}$ simplifies to $ba^2 = a^2b$. Thus we obtain the desired presentation. •

Using GAP[9], we investigated shorter presentations for S_5. We eliminated each of the relators of S_5 and checked whether the obtained presentation could still be a presentation for S_5. To check this, we determined various large p-quotients of the considered group and the abelian invariants of the subgroups of index at most 4. This experimental evidence suggests that the first of the three relators of S_5 in Lemma 13 is redundant and hence we propose the following conjecture.

Conjecture 14 *S_5 has deficiency zero with the presentation*

$$S_5 = \langle a, b \mid a^2/[b, a]^2, (b^2)^{[b,a]}/b^{-2}\rangle.$$

5.2 Higher coclasses and 3-generator groups

It is not difficult to construct further infinite pro-2-groups with finite coclass using the functionality of Gap. For example, every central extension of the cyclic group of order 2 with an infinite pro-2-group of finite coclass is of this type.

We iterated the construction of such central extensions starting with the group S_1. This yields a variety of infinite pro-2-groups with finite coclass and many of these groups have trivial Schur multiplicator. In particular, there are various 3-generator infinite pro-2-groups with trivial Schur multiplicator arising this way: we found 1 such group of coclass 5, 5 such groups of coclass 6 and 26 such groups of coclass 7. A presentation for the 3-generator group of coclass 5 is the following.

$$S_6 := \langle a, t, b, c \mid a^2/b^2, a^8, c^2, t^a/(t^{-1}c), b^a/(bc), b^t/(b^3c), [c, a], [c, b]\rangle.$$

Lemma 15 *S_6 is a 3-generator group of deficiency at least -1 with*

$$S_6 = \langle a, t, b \mid a^2/b^2, [b, a]^2, t^a/(t^{-1}[b, a]), b^t/(aba)\rangle.$$

Proof: The relation $b^a = bc$ implies that $b^2 = (b^2)^a = bcbc$. As $c^2 = 1$, it follows that $[b, c] = 1$. Hence the relator $[b, c]$ is redundant. Also, $b = b^{a^2} = (bc)^a = b^ac^a = bcc^a$ hence $c^a = c^{-1}$ follows. Using $c^2 = 1$, we obtain that $[a, c] = 1$. Hence the relator $[a, c]$ is redundant. Using $t^a = t^{-1}c$, we find that $t^{a^2} = (t^{-1}c)^a = ctc$ and $t^{a^3} = ct^{-1}$. Hence

$t^{a^4} = t$ and $[t, a^4] = [t, b^4] = 1$. Now $b^4 = (b^4)^t = (b^t)^4 = (b^3c)^4 = b^{12}$ and thus $b^8 = 1$ follows. This yields that the relator a^8 is redundant. In summary, we obtain that

$$S_6 = \langle a, t, b, c \mid a^2/b^2, c^2, t^a/(t^{-1}c), b^a/(bc), b^t/(b^3c)\rangle.$$

Now we eliminate c using that $c = [b, a]$ and obtain the desired presentation.
A Smith normal form computation on this presentation of S_6 shows readily that S_6 has the abelian invariants $(2, 2, 2)$ and hence is a 3-generator group. •

Using GAP[9] in the same form as preceding Conjecture 14, we investigated shorter presentations for S_6. Our experimental evidence suggests that the second of the four relators of S_6 in Lemma 15 is redundant and hence we propose the following conjecture.

Conjecture 16 *S_6 has deficiency zero with the presentation*

$$S_6 = \langle a, t, b \mid a^2/b^2, t^a/(t^{-1}[b, a]), b^t/(aba)\rangle.$$

6 Infinite families – the metacyclic case

Now we investigate the infinite families of finite 2-groups arising from the infinite pro-2-groups exhibited in the previous section. We first consider the simplest case: the infinite families associated with S_1, S_2 and S_3. More generally, we consider the infinite pro-2-groups

$$M_n = \langle a, t \mid a^{2^n}, t^a/t^{-1}\rangle.$$

Lemma 17 *Let $\mathcal{G} = (G_0, \ldots)$ be an infinite family associated with M_n for some $n \in \mathbb{N}$. Then each G_j in $\mathcal{G}$ is metacyclic and there exist $h, k \in \mathbb{N}_0$ and $l \in \mathbb{Z}_2$ (the 2-adic numbers) so that*

$$G_j = \langle a, t \mid a^{2^n}/t^{2^{j+h}}, t^a/t^{-1+l2^j}, t^{2^{j+k}}\rangle.$$

Proof: The group M_n has dimension 1. By Theorem 8, this implies that the family $\mathcal{G}$ has a parametrised presentation on generators $X \cup Y$ with $X = \{a, t\}$ and $Y = \{y\}$ and relators $R_j \cup Q_j \cup O$. We discuss these three sets of relators in more detail.
The set R_j contains three relators. Since Y contains one element only, the words $y_i(Y)$ are all just powers of y. The word $t(X)$ is a generator of a suitable translation subgroup T of M_n and hence can be chosen as t^{2^m} for some m. Thus there exist $u, v, w \in \mathbb{Z}_2$ so that the relators in R_j translate to the relations

$$a^{2^n} = y^{u2^j}, \quad t^a = t^{-1}y^{v2^j}, \quad \text{and} \quad t^{2^m} = y^{1+w2^j}.$$

Replacing y^{1+w2^j} by y does not affect the general form of the relations. Hence we can assume that the third relation in R_j reads $t^{2^m} = y$. We use this relation to eliminate y from the generating set and obtain that R_j has the form $R_j = \{a^{2^n}/t^{u2^{j+m}}, t^a/t^{-1+v2^{j+m}}\}$ for some $u, v \in \mathbb{Z}_2$. Let $u = u'2^c$ with u' invertible in $\mathbb{Z}_2$. Then again replacing $t^{u'}$ by t does not affect the general form of the relations. Further let $l = v2^m$ and $h = m + c$. Then we obtain that $R_j = \{a^{2^n}/t^{2^{j+h}}, t^a/t^{-1+l2^j}\}$.

As M_n has dimension 1 for every n, it follows that Q_j contains a single relator only and this has the form $y^{2^{b+j}}$ for some b. The elimination of y translates this to $t^{2^{b+j+m}} = t^{2^{k+j}}$ for $k = m + b$.
It remains to show that the relators in O are redundant. These are two relators of the form $y^a v_{11}$ and $y^t v_{12}$, where the words v_{1j} reflect the action of the point group on the translation subgroup. Since a acts by inverting and t acts trivially on the translations, we obtain that $v_{11} = y$ and $v_{12} = y^{-1}$. The elimination of y translates this to $(t^{2^m})^a = (t^{2^m})^{-1}$ and $(t^{2^m})^t = t^{2^m}$. The later relation is obviously redundant. The first relator follows from Q_j and R_j if $m \geq b$. Note that we can choose m large enough so that this holds and hence both relators in O are redundant. •

A metacyclic group has deficiency zero if and only if it has trivial Schur multiplicator. In this case, a deficiency zero presentation can be read off due to a result by Beyl, see [5], page 66. We observe that we can also read off parametrised presentations for infinite families of metacyclic groups.

Theorem 18 *Let* $G_j = \langle a, t \mid a^{2^n}/t^{2^{j+h}}, t^a/t^{-1+l2^j}, t^{2^{j+k}} \rangle$ *as in Lemma 17.*

a) *Then* $M(G_j) = 1$ *if and only if* $h = k - 1$.
b) *Let* u_j *be the inverse of* $-(1 + l2^{j-1})$ *modulo* 2^{j+k} *and define* $v_j = u_j + 2^{2(j+k)-1}$. *If* $M(G_j) = 1$, *then*

$$G_j \cong \langle a, t \mid a^{2^n}/t^{2^{j+h}}, \left[a, t^{-v_j}\right] = t^2 \rangle.$$

Proof: a) As observed on page 66 in [5], we have that $|M(G_j)| = 2^{j+h} gcd(2^{j+k}, -2 - l2^j/2^{j+k} = 2^{j+h+1-j-k} = 2^{h-k+1}$. This yields a).
b) Theorem 2 on page 66 of [5] yields this result for some integer v_j. Following the proof of this Theorem, we find that $v_j = u_j + 2^{2(j+k)-1}$ as desired. •

Note that this implies that almost all 2-groups with coclass at most 2 and trivial Schur multiplicator are metacyclic.

7 Infinite families – the non-metacyclic case

Our aim in this section is to investigate some examples of infinite families of non-metacyclic groups concerning their deficiencies. We use a combination of theoretical and computational methods.

7.1 Families associated with S_4

We use Theorem 8 to determine parametrised presentations for the infinite families associated with S_4. We determine some of the words needed for Theorem 8 by experimental methods and hence propose the following observation.

Observation 19 *The infinite families associated to* S_4 *are defined by two parameters* (e, f) *and have a parametrised presentation of the form*

$$G_j(e, f) = \langle a, u, y \mid R_j(e, f) \cup Q_j \cup O \rangle$$

with

- $R_j = \{a^2u^{-4}y^{e2^j}, (u^2)^a u^2 y^{f2^j}, (u^{-1}a)^{2^7}y^{-1}\}$,
- $Q_j = \{y^{2^{j+2}}\}$,
- $O = \{y^a y, y^u y\}$.

Proof: Let $X = \{a, u\}$ and $Y = \{y\}$ in Theorem 8. We determine the relators in Theorem 8 more precisely. For this purpose, we have to determine the words $t(X)$, $y_i(Y)$ and $v_{1,h}(Y)$ and a suitable value for b.
The word $t(X)$ corresponds to a module generator for a suitable translation subgroup of S_4. The element $(u^{-1}a)^{2^m}$ is a module generator of the translation subgroup with index 2^{3+m} in S_4. Experimental evidence suggests that $m = 7$ is a suitable value for m. Hence we use $t(X) = (u^{-1}a)^{128}$.
By suitably replacing y by a power of y, we can assume that $y_t = 1$. The relators in O display the action of the generators in X on y. This action is equivalent to the action of S_4 on quotients of its translation subgroup T. Hence the conjugation of a and u both invert y which determines the words $v_{1,h}$. Finally, experimental evidence suggests that $b = 2$ is a suitable value for b in Theorem 8. •

We can simplify the presentation of Observation 19 as follows.

Lemma 20 *In the presentation $G_j(e, f)$ the generator y can be eliminated using the relation $y = (u^{-1}a)^{2^7}$ and the relators in O are redundant. Thus*

$$G_j(e, f) = \langle a, u \mid a^2u^{-4}(u^{-1}a)^{e2^{j+7}}, (u^2)^a u^2 (u^{-1}a)^{f2^{j+7}}, (u^{-1}a)^{2^{j+9}} \rangle$$

and $G_j(e, f)$ has deficiency at least -1.

Proof: The elimination of y is obvious. Further, one of the two relators in O is redundant, since $u^{-1}a$ acts trivially on y. Let $j \geq 7$. Then the following relations remain

$$a^2u^{-4} = (a^{-1}u)^{e2^j},\ (u^2)^a u^2 = (a^{-1}u)^{f2^j},\ (a^{-1}u)^{2^{j+2}},\ ((a^{-1}u)^{2^7})^u = (a^{-1}u)^{-2^7}.$$

We replace a by $x = a^{-1}u$ and translate these relations to

$$ux^{-1}ux^{-1}u^{-4} = x^{e2^j},\ u^2(u^2)^x = x^{f2^j},\ x^{2^{j+2}} = 1,\ (x^{2^7})^u = x^{-2^7}).$$

Now the second relation implies that $(u^2)^{x^2} = x^{-f2^j}u^2x^{f2^j}$ and thus

$$(u^2)^{x^4} = x^{-f2^j}(u^2)^{x^2}x^{f2^j} = x^{-f2^{j+1}}u^2x^{f2^{j+1}}.$$

Hence $(u^2)^{x^8} = x^{-f2^{j+2}}u^2x^{f2^{j+2}}$ or $[x^8,\ u^2] = 1$ follows. Therefore, the second relation implies that $u^2xu^2 = x^{f2^j+1}$. Now the first relation yields $ux^{-1}u = u^4x^{e2^j+1}$ or $x^u = x^{-e2^j-1}u^{-2}$ and hence $(x^2)^u = x^{-e2^{j+1}-1}u^{-2}x^{-1}u^{-2} = x^{-e2^{j+1}-f2^j-2}$. We obtain that $(x^8)^u = ((x^2)^u)^4 = x^{-8}$ and thus the other relator in O is also redundant. •

We now use computational methods and GAP[9] to investigate the presentations of $G_j(e, f)$ further. In particular, we use GAP to solve the isomorphism problem for finite p-groups given by a finite presentation and to determine the Schur multiplicators of finite p-groups. Evaluating the presentations of Lemma 20 for $j \leq 10$ and various small values for e and f then yields the following.

Observation 21 *The values* $(e, f) \in \{(0,0),(2,0),(0,2),(1,0),(1,2)\}$ *yield 5 different (pairwise non-isomorphic) infinite families of 2-groups associated with* S_4*.*

a) *The families with* $(e, f) \in \{(0,2),(1,0),(1,2)\}$ *are families of groups with trivial Schur multiplicators.*

b) *The families with* $(e, f) \in \{(0,0),(2,0)\}$ *are families of groups with Schur multiplicators of order 2 in which the family with* $(e, f) = (0,0)$ *is the main line family.*

This observation directly implies the following.

Remark 22 *Lemma 20 yields efficient presentations for the families with parameters* $(e, f) \in \{(0,0),(2,0)\}$*.*

We investigated the three families with trivial Schur multiplicator further. Again using GAP, we investigated shorter presentations for these families. Our approach for this purpose is quite simple: we consider a fixed group $G_j(e, f)$ and eliminate each relator of this group in turn. We then investigate whether the resulting shorter presentation may still define the same group. For the later purpose we compare the p-quotient and the abelian invariants of the subgroups of low index for the new presentation with G_j. We apply this approach to $G_j(e, f)$ with fixed (e, f) and various small j. Our experimental evidence suggests the following conjecture.

Conjecture 23 *The family defined by the parameters* $(1,2)$ *has deficiency zero with the presentation*

$$G_j = \langle a, u \mid a^2u^{-4}t^{2^j}, (u^2)^a u^2 t^{-2^{j+1}} \rangle,$$

where $t = (u^{-1}a)^{2^7}$*.*

For the other two families with parameters $(1,0)$ and $(0,2)$ we could not identify a deficiency zero presentation with our approach.

7.2 Families associated with S_6

We use a similar approach as in Section 7.1 to determine short presentations for the infinite families of finite 2-groups associated with S_6. We note that all the infinite families associated with S_6 have distance at most 3 from the main line and we apply Theorem 9 in the following.

Observation 24 *The infinite families associated to* S_6 *are defined by four parameters* (e, f, g, h) *and have a parametrised presentation of the form*

$$G_j(e,f,g,h) = \langle a, b, t \mid a^2b^{-2}t^{e2^j}, [b,a]^2t^{f2^j}, t^a(t^{-1}[b,a])^{-1}t^{g2^j}, b^t(b^3[b,a])^{-1}t^{h2^j}, t^{2^{j+3}} \rangle.$$

Proof: Recall the presentation for S_6 in Lemma 15. Further, we note that the members of the lower central series $\gamma_c(S_6)$ are generated by $t^{2^{c-1}}$ for $c \geq 4$. Hence by Lemma 6 the main line groups $S_{6,j} = S_6/\gamma_{j+1}(S_6)$ have a presentation of the form

$$S_{6,j} = \langle a, b, t \mid a^2b^{-2}, [b,a]^2, t^a(t^{-1}[b,a])^{-1}, b^t(b^3[b,a])^{-1}, t^{2^j} \rangle.$$

The groups in a family of distance k from the main line are extensions of $S_{6,j}$ by a cyclic group of order 2^k which is identified with a lower central series quotient of S_6. This yields a presentation of the form

$$\langle a, b, t, s \mid a^2b^{-2}s^e, [b,a]^2s^f, t^a(t^{-1}[b,a])^{-1}s^g, b^t(b^3[b,a])^{-1}s^h, t^{2^j}s^{-1}, s^{2^k} \rangle$$

for certain integers e, f, g and h. Now we can use the relation $s = t^{2^j}$ to eliminate s. Further, we can replace j by $j - (3 - k)$ and adjust the parameters e, f, g and h suitably to obtain the desired presentation for the considered groups. •

Again as in Section 7.1, we determine parameters for the 11 infinite families associated with S_6.

Observation 25 *The values* $(e, f, g, h) \in \{(0,0,0,0), (0,0,0,1), (0,0,1,0), (1,0,0,0), (0,0,2,4), (1,4,0,1), (1,4,0,-1), (1,2,0,-2), (1,-2,0,4), (-1,2,0,4), (1,2,0,2)\}$ *yield 11 different (pairwise non-isomorphic) infinite families of 2-groups associated with* S_6*.*

a) *The families with* $(e, f, g, h) \in \{(1,4,0,1), (1,4,0,-1), (1,2,0,-2), (1,-2,0,4), (-1,2,0,4), (1,2,0,2)\}$ *are families of groups with trivial Schur multiplicators.*

b) *The families with* $(e, f, g, h) \in (0,0,0,0), (0,0,0,1), (0,0,1,0), (1,0,0,0), (0,0,2,4)\}$ *are families of groups with Schur multiplicators of order 2 in which the family with* $(e, f, g, h) = (0,0,0,0)$ *is the main line family.*

Finally, we investigated the families of groups with trivial Schur multiplicator further. Again, we applied the same strategy as in Section 7.1. This strategy proved to be successful in all cases here and we obtain the following conjecture.

Conjecture 26 *The families of* S_6 *with trivial Schur multiplicator are deficiency zero with following presentations*

$$\begin{array}{rcl}
G_j(1,4,0,1) & \cong & \langle a,b,t \mid a^2b^{-2}t^{2^j},\ t^a(t^{-1}[b,a])^{-1},\ b^t(b^3[b,a])^{-1}t^{2^j} \rangle, \\
G_j(1,4,0,-1) & \cong & \langle a,b,t \mid a^2b^{-2}t^{2^j},\ t^a(t^{-1}[b,a])^{-1},\ b^t(b^3[b,a])^{-1}t^{-2^j} \rangle, \\
G_j(1,2,0,-2) & \cong & \langle a,b,t \mid a^2b^{-2}t^{2^j},\ t^a(t^{-1}[b,a])^{-1},\ b^t(b^3[b,a])^{-1}t^{-2^{j+1}} \rangle, \\
G_j(1,-2,0,4) & \cong & \langle a,b,t \mid a^2b^{-2}t^{2^j},\ t^a(t^{-1}[b,a])^{-1},\ b^t(b^3[b,a])^{-1}t^{2^{j+2}} \rangle, \\
G_j(-1,2,0,4) & \cong & \langle a,b,t \mid a^2b^{-2}t^{-2^j},\ t^a(t^{-1}[b,a])^{-1},\ b^t(b^3[b,a])^{-1}t^{2^{j+2}} \rangle, \\
G_j(1,2,0,2) & \cong & \langle a,b,t \mid a^2b^{-2}t^{2^j},\ t^a(t^{-1}[b,a])^{-1},\ b^t(b^3[b,a])^{-1}t^{2^{j+1}} \rangle.
\end{array}$$

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